RAMANASRI

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Real Analysis

Previous year Questions from 2016 to 1992

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2017

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2016

1. For that the function $f:(0,\infty)\to R$ given by $f(x)=x^2\sin\frac{1}{x}, 0<x<\infty$ Show that there is a differentiable function $g:R\to R$ that extends f **[10 marks]**
2. Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:
$x_1=\frac{1}{2}, y_1=1, x_n=\sqrt{x_{n-1}y_{n-1}}, n=2,3,4,\ldots \frac{1}{y_n}=\frac{1}{2}\left(\frac{1}{x_n}+\frac{1}{y_{n-1}}\right), n=2,3,4,\ldots$ and Prove that
$x_{n-1}<x_n<y_n<y_{n-1}, n=2,3,4,\ldots$ and deduce that both the sequence converges to the same $\lim tl$ where $\frac{1}{2}<l<1.$ **[10 marks]**
3. Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1}$ conditionally convergent (if you use any theorem (s) to show it then you must give a proof of that theorem(s). **[15 marks]**
4. Find the relative maximum minimum values of the function $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$ **[15 marks]**
5. Let $f:R\to R$ be a continuous function such $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist and are finite. Prove that is uniformly continuous on ¡ **[15 marks]**

2015

6. Test for convergence $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n}{n^2+1}\right)$ **[10 Marks]**
7. Is the function $f(x)=\begin{cases}\frac{1}{n}, & \frac{1}{n+1}<x\le\frac{1}{n}\\ 0 & x=0\end{cases}$ Riemann Integrable? If yes, obtain the value of $\int_0^1 f(x)dx$ **[15 Marks]**
8. Test the series of functions $\sum_{n=1}^{\infty}\frac{nx}{1+n^2x^2}$ for uniform convergence **[15 Marks]**
9. Find the absolute maximum and minimum values of the function $f(x,y)=x^2+3y^2-y$ over the region $x^2+2y^2\le 1$ **[15 Marks]**

2014

10. Test the convergence of the improper integral $\int_1^{\infty}\frac{dx}{x^2\left(1+e^{-x}\right)}$ **[10 Marks]**
11. Integrate $\int_1^0 f(x)dx,$,where $f(x)=\begin{cases}2x\sin\frac{1}{x}\cos\frac{1}{x}, x\in[0,1]\\ 0 \qquad x=0\end{cases}$ **[15 Marks]**

12. Obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function $f(x,y) = \begin{cases} \frac{xy(3x^2 - 2y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0 & ,(x,y) \neq (0,0) \end{cases}$

Also, discuss the continuity $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ of at $(0,0)$ **[15 Marks]**

13. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$ by the method of Lagrange multiplies. **[15 Marks]**

2013

14. Let $f(x) = \begin{cases} \frac{x^2}{2} + 4 & \text{if } x \geq 0 \\ \frac{-x^2}{2} + 2 & \text{if } x < 0 \end{cases}$ Is f Riemann integrable in the interval $[-1,2]$? Why? Does there exist a function g such that $g'(x) = f(x)$? Justify your answer. **[10 Marks]**

ONE QUESTION MISSING 10 MARKS

15. Show that the series $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n + x^2}$,is uniformly convergent but not absolutely for all real values of x. **[13 Marks]**

16. Show that every open subset of R is countable union of disjoint open intervals **[14 Marks]**

17. Let $[x]$ denote the integer part of the real number x, i.e., if $n \leq x < n+1$ where n is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Riemann integrable in the function in $[-1,2]$? If not, explain why. If it is integrable, compute $\int_{-1}^{2} ([x]^2 + 3) dx$ **[10 Marks]**

2012

18. Let, $f_n(x) = \begin{cases} 0, & \text{if } x < \frac{1}{n+1} \\ \sin\frac{\pi}{x}, & \text{if } x < \frac{1}{n+1} \leq x \leq \frac{1}{n}, \\ 0, & \text{if } x > \frac{1}{n}. \end{cases}$ Show that $f_n(x)$ converges to a continuous function but not uniformly. **[12 Marks]**

19. Show that the series $\sum_{n=1}^{\infty} \left(\frac{\pi}{\pi + 1}\right)^n n^6$ is convergent **[12 Marks]**

20. Let $f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x,y) = (0,0) \end{cases}$ Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$. **[15 Marks]**

21. Find the minimum distance of the line given by the planes $3x+4y+5z=7$ and $x-z=9$ and from the origin, by the method of Lagrange's multipliers. **[15 Marks]**

22. Let $f(x)$ be differentiable on $[0,1]$ such that $f(1)=f(0)=0$ and $\int_0^1 f^2(x)\,dx=1$. Prove that $\int_0^1 xf(x)f'(x)dx=-\frac{1}{2}$ **[15 Marks]**

23. Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify your answer **[20 Marks]**

2011

24. Let $S=(0,1)$ and f be defined by $f(x)=\frac{1}{x}$ where $0<x\le 1$ (in R). Is f uniformly continuous on S? Justify your answer. **[12 Marks]**

25. Let $f_n(x)=nx(1-x)^n, x\in[0,1]$. Examine the uniform convergence of $\{f_n(x)\}$ on $[0,1]$ **[15 Marks]**

26. Find the shortest distance from the origin $(0,0)$ to the hyperbola $x^2+8xy+7y^2=225$ **[15 Marks]**

27. Show that the series for which the sum of first n terms $f_n(x)=\frac{nx}{1+n^2x^2}, 0\le x\le 1$ cannot be differentiated term-by-term at $x=0$. What happens at $x\ne 0$? **[15 Marks]**

28. Show that if $S(x)=\sum_{n=1}^{\infty}\frac{1}{n^3+n^4x^2}$, then its derivative $S'(x)=-2x\sum_{n=1}^{\infty}\frac{1}{n^2\left(1+nx^2\right)^2}$, for all x **[20 Marks]**

2010

29. Discuss the convergence of the sequence $\{x_n\}$ where $X_n=\frac{\sin\left(\frac{n\pi}{2}\right)}{8}$ **[12 Marks]**

30. Define $\{x_n\}$ by $x_1=5$ and $x_{n+1}=\sqrt{4+x_n}$ for $n>1$ Show that the sequence converges to $\left(\frac{1+\sqrt{17}}{2}\right)$ **[12 Marks]**

31. Define the function $f(x)=\begin{cases}x^2\sin\frac{1}{x}, & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$. Find $f'(x)$. Is $f'(x)$ continuous at $x=0$? Justify your answer. **[15 Marks]**

32. Consider the series $\sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^2}$. Find the values of x for which it is convergent and also the sum function. Is the converse uniform? Justify your answer. **[15 Marks]**

33. Let $f_n(x)=x^n$ on $-1<x\le 1$ for $n=1,2.....$ Find the limit function. Is the convergence uniform? Justify your answer. **[15 Marks]**

2009

34. State Roll's theorem. Use it to prove that between two roots of $e^x \cos x = 1$ there will be a root o $e^x \sin x = 1$

2+10=[12 Marks]

35. Let $f(x)\begin{cases} \frac{|x|}{2}+1 & \text{if } x<1 \\ \frac{x}{2}+1 & \text{if } 1 \le x < 2 \\ -\frac{|x|}{2}+1 & \text{if } 2 \le x \end{cases}$ What are the points of discontinuity of f, if any? What are the points where f is not differentiable, if any? Justify yours answer. **[12 Marks]**

36. Show that the series $\left(\frac{1}{3}\right)^2 + \left(\frac{1.4}{3.6}\right)^2 + \ldots.. + \left(\frac{1.4.7......(3n-2)}{3.6.9...........3n}\right)^2 + \ldots...$ converges **[15 Marks]**

37. Show that if $f:[a,b] \to R$ is a continuous function then $f([a,b]) = [c,d]$ form some real numbers c and d, $c \le d.$

[15 Marks]

38. Show that: $\lim\limits_{x \to 1} \sum\limits_{n=1}^{\infty} \frac{n^2x^2}{n^4+x^4} = \sum\limits_{n=1}^{\infty} \frac{n^2}{n^4+1}$ Justify all steps of your answer by quoting the theorems you are using

[15 Marks]

39. Show that a bounded infinite subset R must have a limit point **[15 Marks]**

2008

40. (i) For $x > 0$, show $\frac{x}{1+x} < \log(1+x) < x$ **[6 Marks]**

(ii) Let $T = \left\{\frac{1}{n}, n \in N\right\} \cup \left\{1 + \frac{3}{2n}, n \in N\right\} \cup \left\{6 - \frac{1}{3n}, n \in N\right\}$. Find derived set T of T. Also find Supremum of T and greatest number of T. **[6 Marks]**

41. If $f: R \to R$ is continuous and $f(x+y) = f(x) + f(y),$ for all $x, y \in R$ then show that $f(x) = xf(1)$ for all $x \in R.$

[12 Marks]

42. Discuss the convergence of the series $\frac{x}{2} + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots.., x > 0.$ **[15 Marks]**

43. Show that the series $\sum \frac{1}{n(n+1)}$ is equivalent to $\frac{1}{2}\prod\limits_{2}^{\infty}\left(1 + \frac{1}{n^2-1}\right)$ **[15 Marks]**

44. Let f be a continuous function on $[0,1]$. Using first Mean Value theorem on Integration, prove that

$\lim\limits_{n \to \infty} \int\limits_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0)$ **[15 Marks]**

45. (i) Prove that the sets $A = [0,1],\ B = (0,1)$ are equivalent sets. **[6 Marks]**

(ii) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x},\ x \in \left(0, \frac{\pi}{2}\right)$ **[9 Marks]**

2007

46. Show that the function given by $f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2} & (x,y) \ne (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exists at $(0,0)$ **[12 Marks]**

47. Using Lagrange's mean value theorem, show that $|\cos b - \cos a| \le |b-a|$ **[12 Marks]**

48. Given a positive real number a and any natural number n, prove that there exists one and only one positive real number ξ such that $\xi^n = a$ **[20 Marks]**

49. Find the volume of the solid in the first octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$ **[20 Marks]**

50. Rearrange the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ to converge to 1 **[20 Marks]**

2006

51. Examine the convergence of $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$ **[12 Marks]**

52. Prove that the function f defined by $f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$ is nowhere continuous. **[12 Marks]**

53. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one value $\xi,\ a < \zeta < b$ for which $f''(\zeta) < 0$. **[20 Marks]**

54. Show that the function given by $f(x,y) = \begin{cases} \dfrac{x^3+2y^2}{x^2+y^2}, & (x,y) \ne (0,0) \\ 0 \quad , & (x,y) = (0,0) \end{cases}$ (i) is continuous at $(0,0)$ (ii) possesses partial derivative $f_x(0,0)$ and $f_y(0,0)$ **[20 Marks]**

55. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ **[20 Marks]**

2005

56. If u, v, w are the roots of the equation in λ and $\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$, evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ **[12 Marks]**

57. Evaluate $\iiint \ln(x+y+z)\ dxdydz$ The integral being extended over all positive values of x, y, z such that $x+y+z \le 1$ **[12 Marks]**

58. If f' and g' exist for every $x \in [a,b]$ and if $g'(x)$ does not vanish anywhere (a,b), show that there exists c in (a,b) such that $\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$ **[30 Marks]**

59. Show that $\int_0^{\infty} e^{-t} t^{n-1}\ dt$ is an improper integral which converges for $n > 0$ **[30 Marks]**

2004

60. Show that the function $f(x)$ defined as: $f(x) = \frac{1}{2^n},\ \frac{1}{2^{n+1}} \le x \le \frac{1}{2^n},\ n = 0,1,2,.....$ and $f(0) = 0$ is integrable in $[0,1]$, although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x)dx = \frac{2}{3}$ **[12 Marks]**

61. Show that the function $f(x)$ defined on by: $f(x) = \begin{cases} x & \text{when } x \text{ is irrational} \\ -x & \text{when } x \text{ is rational} \end{cases}$ is continuous only at $x = 0$ **[12 Marks]**

62. If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c. **[20 Marks]**

63. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$ **[20 Marks]**

64. Let $f(x) \geq g(x)$ for every x in $[a,b]$ and f and g are both bounded and Riemann integrable on $[a,b]$. At a point $c \in [a,b]$, let f and g be continuous and $f(c) > g(c)$ then prove that $\int_a^b f(x)\, dx > \int_a^b g(x)\, dx$ and hence show that $-\frac{1}{2} < \int_a^b \frac{x^3 \cos 5x}{2 + x^2} dx < \frac{1}{2}$ **[20 Marks]**

2003

65. Let a be a positive real number and $\{x_n\}$ sequence of rational numbers such that $\lim_{n\to\infty} x_n = 0$ .Show that $\lim_{n\to\infty} a x_n = 1$ **[12 Marks]**

66. If a continuous function of x satisfies the functional equation $f(x+y) = f(x) + f(y)$ then show that $f(x) = \alpha x$ where α is a constant. **[12 Marks]**

67. Show that the maximum value of $x^2 y^2 z^2$ subject to condition $x^2 + y^2 + z^2 = c^2$ is $\frac{c^2}{27}$. Interpret the result **[20 Marks]**

68. The axes of two equal cylinders intersect at right angles. If a be their radius, then find the volume common to the cylinder by the method of multiple integrals. **[20 Marks]**

69. Show that $\int_0^\infty \frac{dx}{1 + x^2 sin^2 x}$ is divergent **[20 Marks]**

2002

70. Prove that the integral $\int_0^\infty x^{m-1} e^{-x}\, dx$ is convergent if and only if $m > 0$. **[12 Marks]**

71. Find all the positive values of a for which the series $\sum_{n=1}^{\infty} \frac{(an)^n}{n!}$ converges. **[12 Marks]**

72. Test uniform convergence of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$, where $p > 0$ **[20 Marks]**

73. Obtain the maxima and minima of $x^2 + y^2 + z^2 - yz - zx - xy$ subject to condition $x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0$ **[25 Marks]**

74. A solid hemisphere H of radius 'a' has density ρ depending on the distance R from the center of and is given by $\rho = k(2a - R)$ where k is a constant. Find the mass of the hemisphere by the method of multiple integrals **[15 Marks]**

2001

75. Show that $\int_0^{\pi/2} \frac{x^n}{\sin^m x} dx$ exists if and only if $m < n + 1$ **[12 Marks]**

76. If $\lim_{n\to\infty} a_n = l$, then prove that $\lim_{n\to\infty} \frac{a_1 + a_2 + \ldots a_n}{n} = l$, **[12 Marks]**

77. A function f is defined in the interval (a,b) as follows

$$f(x)=\begin{cases}\frac{1}{q^2} \text{ when } x=\frac{p}{q}\\ \frac{1}{q^3} \text{ when } x=\sqrt{\frac{p}{q}}\end{cases}$$ where p,q relatively prime integers. $f(x)=0$ for all other values of x. Is f Riemann integrable? Justify your answer. **[20 Marks]**

78. Show that $U=xy+yz+zx$ has a maximum value when the three variables are connected by the relation $ax+by+cz=1$ and a,b,c are positive constants satisfying the condition $2(ab+bc+ca)>(a^2+b^2+c^2)$ **[25 Marks]**

79. Evaluate $\iiint\left(ax^2+by^2+cz^2\right)\,dxdydz$ taken throughout the region $x^2+y^2+z^2\le R^2$ **[15 Marks]**

2000

80. Given that the terms of a sequence $\{a_n\}$ are such that each $a_k,\ k\le 3,$ is the arithmetic mean of its two immediately preceding terms. Show that the sequence converges. Also find the limit of the sequence. **[12 Marks]**

81. Determine the values of x for which the infinite product $\prod_{n=0}^{\infty}\left(1+\frac{1}{x^{2n}}\right)$ converges absolutely. Find its value whenever it converges. **[12 Marks]**

82. Suppose f is twice differentiable real valued function in $(0,\infty)$ and $M_{0,}M_1$ and M_2 the least upper bounds of $|f(x)|,\ |f'(x)|$ and $|f''(x)|$ respectively in $(0,\infty)$. Prove for each $x>0,\ h>0$ that $f'(x)\frac{1}{2h}\left[f(x+2h)-f(x)\right]-hf'(u)$ for some $u\in(x,x+2h)$. Hence show that $\mathrm{M}_1{}^2\le 4\mathrm{M}_0\mathrm{M}_{2.}$ **[20 Marks]**

83. Evaluate $\iint_S\left(x^3dydz+x^2ydzdx+x^2zdxdy\right)$ by transforming into triple integral where S is the closed surface formed by the cylinder $x^2+y^2=a^2, 0\le z\le b$ and the circular disc $x^2+y^2\le a^2, z=0$ and $x^2+y^2\le a^2, z=b$ **[20 Marks]**

1999

84. Let A be a subset of the metric space (M,ρ). If (A,ρ) is compact, then show that A is a closed subset of (M,ρ) **[20 Marks]**

85. A sequence $\{S_n\}$ is defined by the recursion formula $\mathrm{S}_{n+1}=\sqrt{3S_n}, S_1=1$. Does this sequence converge? If so, find $\lim S_n$ **[20 Marks]**

86. Test for convergence the integral $\int_0^1 x^p\left(\log\frac{1}{x}\right)^q dx$ **[20 Marks]**

87. Find the shortest distance from the origin to the hyperbola $x^2+8xy+7y^2=225,\ z=0$ **[20 Marks]**

88. Show that the double integral $\iint_R\frac{x-y}{(x+y)^3}\,dxdy$ does not exist over $R=[0,1;0,1]$ **[20 Marks]**

89. Verify the Gauss divergence theorem for $\overline{F}=4x\hat{e}_x-2y^2\hat{e}_y+z^2\hat{e}_z$ taken over the region bounded by $x^2+y^2=4,\ z=0$ and $z=3$ where $\hat{e}_{x,}\hat{e}_{y,}\hat{e}_z$ are unit vectors along $x-, y-$ and $z-$directions respectively. **[20 Marks]**

1998

90. Let X be a metric space and $E \subset X$. Show that

(i) Interior of E is the largest open set contained in E

(ii) Boundary of E =(closure of E) $\bigcap$ (closure of $X-E$) **[20 Marks]**

91. Let (X,d) and (Y,e) be metric spaces with X compact and $f: X \to Y$ continuous. Show that f is uniformly continuous. **[20 Marks]**

92. Show that the function $f(x,y) = 2x^4 - 3x^2y + y^2$ has $(0,0)$ as the only critical point but the function neither has a minima nor maxima at $(0,0)$ **[20 Marks]**

93. Test the convergence of the integral $\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx, \ \ a \geq 0$ **[20 Marks]**

94. Test the series $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$ for uniform convergence. **[20 Marks]**

95. Let $f(x) = x$ and $g(x) = x^2$. Does $\int_0^1 fog$ exist? If it exists then find its value **[20 Marks]**

1997

96. Show that a non-empty set P in R^n each of whose points is a limit-point is uncountable. **[20 Marks]**

97. Show that $\iiint_D xyz \ dxdydz = \frac{a^2b^2c^2}{6}$ where domain D is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ **[20 Marks]**

98. If $u = \sin^{-1}\left[\left(x^2 + y^2\right)^{1/5}\right]$, Prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2}{25} \tan u (2\tan^2 u - 3)$ **[20 Marks]**

1996

99. Let F be the set of all real valued bounded continuous functions defined on the closed interval $[0,1]$. Let d be a mapping of $F \times F$ into $R,$ the set of real numbers, defined by $d(f,g) = \int_0^1 |f(x) - g(x)| \ dx \ \ \forall f,g \in F$. Verify that d is a metric for F **[20 Marks]**

100. Prove that a compact set in a metric space is a closed set. **[20 Marks]**

101. Let $C[a,b]$ denote the set of all functions f on $[a,b]$ which have continuous derivatives at all points of $I = [a,b]$. For $f,g \in C[a,b]$ define $d(f,g) = |f(a) - g(b)| + \sup\{|f'(x) - g'(x)|, x \in I\}$. Show that the space $(C[a,b],d)$ is a complete **[20 Marks]**

102. A function f is defined in the interval (a,b) as follows:

$$f(x) = \begin{cases} q^{-2} & \text{when } x = pq^{-1} \\ q^{-3} & \text{when } x = \left(pq^{-1}\right)^{1/2} \end{cases}$$

where p,q are relatively prime integers; $f(x) = 0,$ for all other values of x. Is f Riemann integrable? Justify your answer. **[20 Marks]**

103. Test for uniform convergence, the series $\sum_{n=1}^{\infty} \frac{2^n x^{2n-1}}{1+x^{2n}}$ **[20 Marks]**

104. Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y) \ dxdy$ **[20 Marks]**

1995

1. Let K and F be nonempty disjoint closed subjects of R^2. If K is bounded, show that there exists $\delta > 0$ such that $d(x,y) \geq \delta$ for $x \in K$ and $y \in F$ where $d(x,y)$ is the usual distance between x and y. **[20 Marks]**
2. Let f be a continuous real function on R such that f maps open interval into open intervals. Prove that f is monotonic. **[20 Marks]**
3. Let $c_n \geq 0$ for all positive integers n such that is convergent. Suppose $\{S_n\}$ is a sequence of distinct points in (a,b) For $x \in [a,b]$, define $\alpha(x) = \sum c_n \ \{n : x > S_n\}$. Prove that α is an increasing function. If f a continuous real function on $[a,b]$, show that $\int_a^b f d\alpha = \sum c_n f(S_n)$ **[20 Marks]**
4. Suppose f maps an open ball $U \subset R^n$ into R^m and f is differentiable on U. Suppose there exists a real number $M > 0$ such that $\|f(x)\| \leq M \ \ \forall x \in U$. Prove that $|f(b) - f(a)| \leq M|b-a| \ \forall a,b \in U$ **[20 Marks]**
5. Find and classify the extreme values of the function $f(x,y) = x^2 + y^2 + x + y + xy$ **[20 Marks]**
6. Suppose α is real number not equals to $n\pi$ for any integer n. Prove that $\int_0^\infty \int_0^\infty e^{-(x^2+2xy\cos\alpha + y^2)} \ dxdy = \dfrac{\alpha}{2\sin\alpha}$ **[20 Marks]**

1994

7. Examine the (i) absolute convergence (ii) uniform convergence of the series $(1-x) + x(1-x) + x^2(1-x) + \ldots$ in $[-c,1], \ 0 < c < 1$ **[20 Marks]**
8. Prove that $S(x) = \sum \dfrac{1}{n^p + n^q x^2}$, $p > 1$ is uniformly convergent for all values of x and can be differentiate term by term if $q < 3p < 2$ **[20 Marks]**
9. Let the function f be defined on $[0,1]$ by the condition $f(x) = 2rx$ when $\dfrac{1}{r+1} < x < \dfrac{1}{r}$, $r > 0$ Show that f is Riemann integrable in $[0,1]$ and $\int_0^1 f(x) \ dx = \dfrac{\pi^2}{6}$ **[20 Marks]**
10. By means of substitution $x+y+z = u, y+z = uv, z = uvw$ evaluate $\iiint (x+y+z)^n xyz \ dxdydz$ taken over the volume bounded by $x=0, y=0, z=0, x+y+z=1$ **[20 Marks]**

1993

11. Examine for Riemann integrability over $[0,2]$ of the function defined in $[0,2]$ by

$$f(x) = \begin{cases} x + x^2, & \text{for rational valuesof } x \\ x^2 + x^3, & \text{for irrational values of } x \end{cases}$$

[20 Marks]

12. Prove that $\int_0^\infty \dfrac{\sin x}{x} \ dx$ converges and conditionally converges. **[20 Marks]**
13. Evaluate $\iiint \dfrac{dxdydz}{x+y+z+1}$ over the volume bounded by the coordinate planes and the plane $x+y+z=1$ **[20 Marks]**

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14. If we metrize the space of functions continuous on $[a,b]$ by taking $p(x,y)=\sqrt{\int_a^b [x(t)-y(t)]^2\, dt}$ then show that the resulting metric space is NOT complete **[20 Marks]**

15. Examine $2xyz-4zx-2yz+x^2+y^2+z^2-2x-4y-4z$ for extreme values **[20 Marks]**

16. If $U_n=\frac{1+nx}{ne^{nx}}-\frac{1+(n+1)x}{(n+1)e^{(n+1)x}},\ 0<x<1$ Prove that $\frac{d}{dx}\sum U_n=\sum\frac{d}{dx}U_n$ Is the series uniformly convergent in $(0,1)$? Justify your claim. **[20 Marks]**

17. Find the upper and lower Riemann integral for the function defined in the interval $[0,1]$ as follows $\begin{cases}\sqrt{1-x^2} & \text{when } x \text{ is rational}\\ 1-x & \text{when } x \text{ is irrational}\end{cases}$ and show that is NOT Riemann integrable in $[0,1]$. **[20 Marks]**

18. Discuss the convergence or divergence of $\int_0^\infty \frac{x^\beta}{1+x\alpha\sin^2 x}dx,\ \alpha>\beta>0$ **[20 Marks]**

19. Evaluate $\iint\sqrt{\frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2+b^2x^2+a^2y^2}}\ dxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ **[20 Marks]**