RAMANASRI
IAS/IFoS Institute

Complex Analysis

Previous year Questions from 2017 to 1992

Ramanasri Institute

2017

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2017

1. Determine all entire functions $f(z)$ such that 0 is removable singularity of $f\left(\frac{1}{z}\right)$. **[10 Marks]**

2. Using contour integral method, proves that $\int_0^\infty \frac{x\sin mx}{a^2+x^2}dx = \frac{\pi}{2}e^{-ma}$. **[15 Marks]**

3. Let $f = u + iv$ be analytic function on the unit disc $D = \{z \in C : |z| < 1\}$.

   Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ at all points of D. **[15 Marks]**

4. For a function $f : C \to C$ and $n \ge 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)} = f$ Let f be an entire function such that for some $n \ge 1$, $f^{(n)}\left(\frac{1}{k}\right) = 0$ for all $k = 1,2,3,..$ show that f is a polynomial. **[15 Marks]**

2016

5. 2016 Is $v(x, y) = x^3 - 3xy^2 + 2y$ a harmonic function? Prove your claim, if yes find its conjugate harmonic function and hence obtain the analytic function $u(x, y)$ whose real and imaginary parts are u and v respectively **(10 marks)**

6. Let $\gamma;[0,1] \to C$ be the curve $\gamma(t) = e^{2\pi it}, 0 \le t \le 1$ find giving justification the values of the contour integral $\int_\gamma \frac{dz}{4z^2-1}$ **(15 marks)**

7. Prove that every power series represents an analytic function inside its circle of convergence **(20 marks)**

2015

8. Show that the function $v(x, y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also, find the corresponding analytic function $f(z) = u + iv$ in terms of z **(10 Marks)**

9. Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \frac{2z-3}{z^2-3z+2}$ about the point $z = 0$ **(20 Marks)**

10. State Cauchy's residue theorem. Using it, evaluate the integral $\int_C \frac{e^z+1}{z(z+1)(z-i)^2}\,dz;\ C : |z| = 2$ **(15 Marks)**

2014

11. ~~Prove that the function $f(z) = u + iv$, where $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \ne 0$; $f(0) = 0$ satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist. **(10 Marks)**~~

12. Expand in Laurent series the function $f(z) = \frac{1}{z^2(z-1)}$ about $z = 0$ and $z = 1$. **(10 Marks)**

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13. Evaluate the integral $\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues. **(20 Marks)**

2013

14. Prove that if $be^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - be^z$ has n zeros in the unit circle. **(10 Marks)**

15. Using Cauchy's residue theorem, evaluate the integral $I = \int_0^{\pi} \sin^4\theta \, d\theta$ **(15 Marks)**

2012

16. Show that the function defined by $f(z) = \begin{cases} \frac{x^3 y^5 (x+iy)}{x^6 + y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin. **(12 Marks)**

17. Use Cauchy integral formula to evaluate $\int_c \frac{e^{3z}}{(z+1)^4} \, dz$ where c is the circle $|z| = 2$ **(15 Marks)**

18. Expand the function $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent series valid for

(i) $1 < |z| < 3$

(ii) $|z| > 3$

(iii) $0 < |z+1| < 2$

(iv) $|z| < 1$ **(15 Marks)**

19. Evaluate by contour integration $l = \int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$ $a^2 < 1$ **(15 Marks)**

2011

20. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition, $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$ **(12 Marks)**

21. If the function $f(z)$ is analytic and one valued in $|z-a| < R$, prove that for $0 < r < R$,

$f'(a) = \frac{1}{\pi r} \int_0^{2\pi} P(\theta) e^{-i\theta} d\theta$, where $P(\theta)$ is the real part of $f(a + re^{i\theta})$ **(15 Marks)**

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22. Evaluate by Contour integration, $\int_0^1 \frac{dx}{\left(x^2 - x^3\right)^{\frac{1}{3}}}$ **(15 Marks)**

23. Find the Laurent series for the function $f(z) = \frac{1}{1-z^2}$ with centre $z = 1$ **(15 Marks)**

2010

24. Show that $u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of $u(x, y)$. Hence find the analytic function f for which $u(x, y)$ is the real part. **(12 Marks)**

25. (i) Evaluate the line integral $\oint_C f(z)dz$ where $f(z) = z^2$, c is the boundary of the triangle with vertices $A(0,\ 0), B(1,\ 0), C(1,\ 2)$ in that order.

(ii) Find the image of the finite vertical strip $R: x = 5$ to $x = 9,\ -\pi \le \gamma \le \pi$ of z- plane under exponential function **(15 Marks)**

26. Find the Laurent series of the function

$f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right]$ as $\sum_{n=-\infty}^{\infty} C_n z^n$ for $0, |z| < \infty$ where $C_n = \int_0^{\pi} \cos(n\phi - \lambda \sin\phi)d\phi,$

$n = 0, \pm1, \pm2, \ldots$ with λ a given complex number and taking the unit circle C given by $z = e^{i\phi}$ $(-\pi \le \phi \le \pi)$ as contour in this region. **(15 Marks)**

2009

27. Let $f(z) = \frac{a_0 + a_1 \ldots\ldots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \ldots\ldots + b_n z^n}$, $b_n \ne 0$. Assume that the zeros of the denominator are simple.

Show that the sum of the residues of $f(z)$ at its poles is equal to $\frac{a_n - 1}{b_n}$. **(12 Marks)**

28. If α, β, γ are real numbers such that $\alpha^2 > \beta^2 + \gamma^2$ show that :

$$\int_0^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}$$ **(30 Marks)**

2008

29. Find the residue of $\frac{\cot z \coth z}{z^3}$ at $z = 0$ **(12 Marks)**

30. Evaluate $\int_c \left[\frac{e^{2z}}{z^2(z^2 + 2z + 2)} + \log(z - 6) + \frac{1}{(z-4)^2}\right]dz$ where C is the circle $|z| = 3$. State the theorems you use in evaluating above integral **(15 Marks)**

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31. Prove that the function f defined by

$$f(z)=\begin{cases}\dfrac{z^5}{|z|^4} & z\neq 0\\ 0 & z=0\end{cases}$$ is not differentiable at $z=0$ **(12 Marks)**

32. Evaluate (by using residue theorem) $\int_0^{2n}\dfrac{d\theta}{1+8\cos^2\theta}$ **(15 Marks)**

33. Show that the transformation $w=z^2$ is conformal at point $z=1+i$ by finding the images of the lines $y=x$ and $x=1$ which intersect at $z=1+i$ **(15 Marks)**

2006

34. Determine all bilinear transformation which map the half plane $\text{Im}(z)\geq 0$ into the unit circle $|w|\leq 1$ **(12 Marks)**

35. With the aid of residues, evaluate $\int_0^{\pi}\dfrac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta,\ -1<a<1$ **(15 Marks)**

36. Prove that all the roots of $z^7-5z^3+12=0$ lie between the circles $|z|=1$ and $|z|=2$ **(15 Marks)**

2005

37. If $f(z)=u+i\ v$ is an analytic function of the complex variable z and $u-v=e^x(\cos y-\sin y)$, determined $f(z)$ in terms of z. **(12 Marks)**

~~38. Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for~~

~~(i) $1<|z|<3$~~

~~(ii) $|z|<3$ and~~

~~(iii) $|z|<1$~~ ~~**(30 Marks)**~~

2004

39. Find the image of the line $y=x$ under the mapping $w=\dfrac{4}{z^2+1}$ and draw the same. Find the points where this transformation ceases to be conformal. **(12 Marks)**

40. If all zeros of a polynomial $P(z)$ lies in a half plane then show that zeros of the derivatives $P'(z)$ also lie in the same half plane. **(15 Marks)**

~~41. Using contour integration evaluate $\int_0^{2\pi}\dfrac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}d\theta,\ 0<p<1$~~ ~~**(15 Marks)**~~

2003

42. Determine all the bilinear transformations which transform the unit circle $|z|\leq 1$ into the unit circle $|w|\leq 1$ **(12 Marks)**

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43. Discuss the transformation $W = \left(\frac{z - ic}{z + ic}\right)^2$ (c real) showing that the upper half of the W- plane corresponds to the interior of the semi circle lying to the right of imaginary axis in the z- plane. **(15 Marks)**

44. Use the method of contour integration to prove that $\int_0^{\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1 + a^2}}$ $(a > 0)$ **(15 Marks)**

2002

45. Suppose that f and g are two analytic functions on the set ϕ of all complex numbers with $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for $n = 1, 2, 3, \ldots$. Then show that $f(z) = g(z)$ for each z in ϕ **(12 Marks)**

46. (i) Show that, when $0 < |z - 1| < 2,$ that function $f(z) = \frac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of $(z - 1)$ as $\frac{-1}{2(z-1)} - 3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}$ **(15 Marks)**

47. Establish, by contour integration, $\int_0^{\infty} \frac{\cos(ax)}{x^2 + 1} dx = \frac{\pi}{2} e^{-a}$ where $a \geq 0.$ **(15 Marks)**

2001

48. Prove that the Riemann zeta function ζ defined by $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges for $\operatorname{Re} z > 1$ and converges uniformly for $\operatorname{Re} z \geq 1 + \varepsilon$ where $\varepsilon > 0$ is arbitrary small. **(12 Marks)**

49. (i) Find the Laurent series for the function $e^{1/z}$ in $0 < z < \infty.$ Using this expansion, show that $\frac{1}{\pi}\int_0^{\pi} \exp(\cos\theta)\cos(\sin\theta - n\theta) d\theta = \frac{1}{n!}$ for $n = 1, 2, 3, \ldots$ **(15 Marks)**

(ii) Show that $\int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx = \frac{\pi}{\sqrt{2}}$ **(15 Marks)**

2000

50. Show that any four given points of the complex plane can be carried by a bilinear transformation to positions 1, -1, k and –k where the value of k depends on the given points. **(12 Marks)**

51. Suppose $f(\zeta)$ is continuous on a circle C . Show that $\int_C \frac{f(\zeta) d\zeta}{f(\zeta - x)},$ as z varies inside of $C,$ is differentiable under the integral sign. Find the derivative. Hence or otherwise, derive an integral representation for $f'(z)$ if $f(z)$ is analytic on and inside C . **(30 Marks)**

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52. Examine the nature of the function

$$f(z)=\frac{x^2y^5(x+iy)}{x^4+y^{10}}\ ,z\neq 0,$$

$$f(0)=0$$

In a region including the origin and hence show that Cauchy-Riemann equations are satisfied at the origin but $f(z)$ is not analytic there. **(20 Marks)**

53. For the function $f(z)=\frac{-1}{z^3-3z+2}$ find the Laurent series for the domain

(i) $1<|z|<2,$

(ii) $|z|>2.$

Show further that $\oint_c f(z)dz=0$ where C is any closed contour enclosing that points $z=1$ and $z=2$.

(20 Marks)

54. Show that the transformation $w=\frac{2z+3}{z-4}$ transforms the circle $x^2+y^2-4x=0$ into the straight line $4u+3=0,$ where $w=u+iv$. **(20 Marks)**

55. Use Residue theorem show that $\int_{-\infty}^{\infty}\frac{x\sin ax}{x^4+4}dx=\frac{\pi}{2}e^{-a}sina,\ (a>0)$ **(20 Marks)**

56. The function $f(z)$ has a double pole at $z=0$ with residue 2, a simple pole at $z=1$ with residue 2, is analytic at all other finite points of the plane and is bounded as $|z|\to\infty$. If $f(2)=5$ and $f(-1)=2$ find $f(z)$. **(20 Marks)**

57. What kind of singularities the following functions have?

(i) $\frac{1}{1-e^z}$ at $z=2\pi i$

(ii) $\frac{1}{\sin z-\cos z}$ at $z=\frac{\pi}{4}$

(iii) $\frac{\cot\pi z}{(z-a)^2}$ at $z=a$ and $z=\infty.$

In case (iii) above what happens when a is an integer (including $a=0$)? **(20 Marks)**

1998

58. Show that the function

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\ z\neq 0$$

$$f(0)=0$$

is continuous and $C-R$ conditions are satisfied at $z=0,$ but $f'(z)$ does not exist at $z=0$

(20 Marks)

59. Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z=-2$. Specify the region of convergence and the nature of singularity at $z=-2$ **(20 Marks)**

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60. By using the integral representation of $f^n(0)$, prove that $\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\oint_c \frac{x^n e^{xz}}{\lfloor n z^{n+1}} dz,$ where C is any closed contour surrounding the origin. Hence show that $\sum_{n=0}^{\infty}\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta} d\theta$ **(20 Marks)**

61. Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2.$ **(20 Marks)**

62. By integrating round a suitable contour show that $\int_0^{\infty}\frac{x\sin mx}{x^4 + a^4}dx = \frac{\pi}{4b^2}e^{-mb}\sin mb,$ where $b = \frac{a}{\sqrt{2}}$ **(20 Marks)**

63. Using residue theorem, evaluate $\int_0^{2\pi}\frac{d\theta}{3 - 2\cos\theta + \sin\theta}$ **(20 Marks)**

1997

64. Prove that $u = e^x(x\cos y - y\sin y)$ is harmonic and find the analytic function whose real part is u **(20 Marks)**

65. Evaluate $\oint_C \frac{dz}{z+2}$ where C is the unit circle. Deduce that $\int_0^{2\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$ **(20 Marks)**

66. If $f(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^n}$ find residue at a for $\frac{f(z)}{z-b}$ where $A_1, A_2, \ldots, A_n$, a and b are constants. What is the residue at infinity? **(20 Marks)**

67. Find the Laurent series for the function $e^{1/z}$ in $0 < |z| < \infty$. Deduce that $\frac{1}{\pi}\int_0^{\pi}\exp(\cos\theta).\cos(\sin\theta - n\theta)\ d\theta = \frac{1}{n!},\ (n = 0,1,2,\ldots)$ **(20 Marks)**

68. Integrating e^{-z^2} along a suitable rectangular contour show that $\int_0^{\infty} e^{-x^2}\cos 2bx\ dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$ **(20 Marks)**

69. Find the function $f(z)$ analytic within the unit circle, which takes the values $\frac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}$, $0 \le \theta \le 2\pi$ on the circle. **(20 Marks)**

1996

70. Sketchy the ellipse C described in the complex plane by $Z = A\cos\lambda t + iB\sin\lambda t,\ A > B,$ where t is real variable and A, B, λ are positive constants. If C is the trajectory of a particle with $z(t)$ as the position vector of the particle at time $t,$ identify with justification
(i) The two positions where the acceleration is maximum, and
(ii) The tow positions were the velocity in minimum **(20 Marks)**

71. Evaluate $\lim_{z\to 0}\frac{1-\cos z}{\sin(z^2)}$ **(20 Marks)**

72. Show that $z = 0$ is not a branch point for the function $f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity? **(20 Marks)**

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73. Prove that every polynomial equation $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0,\ a_n \neq 0,\ n \geq 1$ has exactly n roots **(20 Marks)**

74. By using residue theorem, evaluate $\int_0^\infty \frac{\log_e(x^2+1)}{x^2+1}\, dx$ **(20 Marks)**

75. About the singularity $z = -2$, find the Laurent expansion of $(z-3)\sin\left(\frac{1}{z+2}\right)$. Specify the region of convergence and the nature of singularity at $z = -2$ **(20 Marks)**

1995

76. Let $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$. Prove that u is a harmonic function. Find a harmonic function v such that $u + iv$ is an analytic function of z. **(20 Marks)**

77. Find the Taylor series expansion of the function $f(z) = \frac{z}{z^2+9}$ around $z = 0$. Find also the radius of convergence of the obtained series. **(20 Marks)**

78. Let C be the circle $|z| = 2$ described counter clockwise. Evaluate the integral $\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$ **(20 Marks)**

79. Let $a \geq 0$. Evaluate the integral $\int_0^\infty \frac{\cos ax}{x^2+1}\, dx$ with the aid of residues **(20 Marks)**

80. Let f be analytic in the entire complex plane. Suppose that there exist a constant $A > 0$ such that $|f(z)| \leq A|z|$ for all z. Prove that there exists a complex number a such that $f(z) = az$ for all z **(20 Marks)**

81. Suppose a power series $\sum_{n=0}^{\infty} a_n z^n$ convergent at a point $z_0 \neq 0$. Let z_1 be such that $|z_1| < |z_0|$ and $z_1 \neq 0$. Show that the series converges uniformly in the disc $\{z : |z| \leq |z_1|\}$ **(20 Marks)**

1994

82. Suppose that z is the position vector of a particles moving on the ellipse $C:\ z = a\cos\omega t + ib\sin\omega t$. Where a, b, ω are positive constants, $a > b$ and t is the time. Determine where
    (i) The velocity has the greatest magnitude.
    (ii) The acceleration has the least magnitude. **(20 Marks)**

83. How many zeros does the polynomial $p(z) = z^4 + 2z^3 + 3z + 4$ possess in (i) the first quadrant, (ii) the fourth quadrant **(20 Marks)**

84. Test of uniform convergence in the region $|z| \leq 1$ the series $\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$ **(20 Marks)**

85. Find Laurent series for
    (i) $\frac{e^{2z}}{(z-1)^3}$ about $z = 1$,
    (ii) $\frac{1}{z^2(z-3)^2}$ about $z = 3$ **(20 Marks)**

86. Find the residue of $f(z) = e^z \operatorname{cosec}^2 z$ at all its poles in the finite plane. **(20 Marks)**

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87. By means of contour integration, evaluate $\int_0^\infty \frac{(\log_e u)^2}{u^2+1}\, du$ **(20 Marks)**

1993

88. In the finite z- plane, show that the function $f(z) = \sec\left(\frac{1}{z}\right)$ has infinitely many isolated singularities in a finite intervals which includes 0. **(20 Marks)**

89. Find the orthogonal trajectories of the family of curves in the xy- plane defined by $e^{-x}(x\sin y - y\cos y) = \alpha$ where α is real function **(20 Marks)**

90. Prove that (by applying Cauchy Integral formula or otherwise) $\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{1.3.5...(2n-1)}{2.4.6...2n} 2\pi$ where $n = 1,2,3...$ **(20 Marks)**

91. If c is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points $(1,\ 1)$ and $(2,\ 3)$ find the value of $\int_c (12z^2 - 4iz)\, dz$ **(20 Marks)**

92. Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges absolutely for $|z| \le 1$ **(20 Marks)**

93. Evaluate $\int_0^\infty \frac{dx}{x^6+1}$ by choosing an appropriate contour **(20 Marks)**

1992

94. If $u = e^{-x}(x\sin y - y\cos y)$, find v such that $f(z) = u + iv$ is analytic. Also find $f(z)$ explicitly as function of z **(20 Marks)**

95. Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$ and let $z = er^{i\theta}$ be any point inside C. Prove that $f(er^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2) f(\mathrm{Re}^{i\phi})}{R^2 - 2Rr\cos(\theta+\phi) + r^2} d\phi$ **(20 Marks)**

96. Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circle $|z| = 1$ and $|z| = 2$. **(20 Marks)**

97. Find the region of convergence of the series whose n th term is $\frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$ **(20 Marks)**

98. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for

(i) $|z| > 3$

(ii) $1 < |z| < 3$

(iii) $|z| < 1$ **(20 Marks)**

99. By integrating along a suitable contour evaluate $\int_0^8 \frac{\cos mx}{x^2+1}\, dx$ **(20 Marks)**

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