

Real Analysis

Previous year Questions from
2017 to 1992

Ramanasri Institute

2017

1. Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}$, $n = 1, 2, 3, \dots$ show that the sequence x_1, x_2, x_3, \dots is convergent. **[10 Marks]**
2. Find the Supremum and the Infimum of $\frac{x}{\sin x}$ on the interval $\left(0, \frac{\pi}{2}\right]$. **[10 Marks]**
3. Let $f(t) = \int_0^t [x] dx$ where $[x]$ denote the largest integer less than or equal to x
 - (i) Determine all the real numbers t at which f is differentiable.
 - (ii) Determine all the real numbers t at which f is continuous but not differentiable. **[15 Marks]**
4. Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100. **[20 Marks]**

2016

5. For that the function $f : (0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = x^2 \sin \frac{1}{x}$, $0 < x < \infty$ Show that there is a differentiable function $g : \mathbb{R} \rightarrow \mathbb{R}$ that extends f **[10 marks]**
6. Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:
 $x_1 = \frac{1}{2}, y_1 = 1, x_n = \sqrt{x_{n-1}y_{n-1}}, n = 2, 3, 4, \dots, \frac{1}{y_n} = \frac{1}{2} \left(\frac{1}{x_n} + \frac{1}{y_{n-1}} \right), n = 2, 3, 4, \dots$ and Prove that $x_{n-1} < x_n < y_n < y_{n-1}, n = 2, 3, 4, \dots$ and deduce that both the sequence converges to the same limit l where $\frac{1}{2} < l < 1$. **[10 marks]**
7. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ conditionally convergent (if you use any theorem (s) to show it then you must give a proof of that theorem(s)). **[15 marks]**
8. Find the relative maximum minimum values of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ **[15 marks]**
9. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on \mathbb{R} **[15 marks]**

2015

10. Test for convergence $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n^2 + 1} \right)$ **[10 Marks]**
11. Is the function $f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \leq \frac{1}{n} \\ 0 & x = 0 \end{cases}$ Riemann Integrable? If yes, obtain the value of $\int_0^1 f(x) dx$ **[15 Marks]**
12. Test the series of functions $\sum_{n=1}^{\infty} \frac{nx}{1+n^2x^2}$ for uniform convergence **[15 Marks]**

13. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 3y^2 - y$ over the region $x^2 + 2y^2 \leq 1$ [15 Marks]

2014

14. Test the convergence of the improper integral $\int_1^{\infty} \frac{dx}{x^2(1+e^{-x})}$ [10 Marks]

15. Integrate $\int_1^0 f(x) dx$, where $f(x) = \begin{cases} 2x \sin \frac{1}{x} \cos \frac{1}{x}, & x \in [0, 1] \\ 0 & x = 0 \end{cases}$ [15 Marks]

16. Obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function $f(x, y) = \begin{cases} \frac{xy(3x^2 - 2y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ [15 Marks]

Also, discuss the continuity $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ of at $(0, 0)$ [15 Marks]

17. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$ by the method of Lagrange multipliers. [15 Marks]

2013

18. Let $f(x) = \begin{cases} \frac{x^2}{2} + 4 & \text{if } x \geq 0 \\ -\frac{x^2}{2} + 2 & \text{if } x < 0 \end{cases}$ Is f Riemann integrable in the interval $[-1, 2]$? Why? Does there exist a function g such that $g'(x) = f(x)$? Justify your answer. [10 Marks]

19. Show that the series $\sum_1^{\infty} \frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent but not absolutely for all real values of x . [13 Marks]

20. Show that every open subset of R is countable union of disjoint open intervals [14 Marks]

21. Let $[x]$ denote the integer part of the real number x , i.e., if $n \leq x < n+1$ where n is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Riemann integrable in the function in $[-1, 2]$? If not, explain why. If it is integrable, compute $\int_{-1}^2 ([x]^2 + 3) dx$ [10 Marks]

2012

22. Let, $f_n(x) = \begin{cases} 0, & \text{if } x < \frac{1}{n+1} \\ \sin \frac{\pi}{x}, & \text{if } x < \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0, & \text{if } x > \frac{1}{n}. \end{cases}$ Show that $f_n(x)$ converges to a continuous function but not uniformly. [12 Marks]

23. Show that the series $\sum_{n=1}^{\infty} \left(\frac{\pi}{\pi+1}\right)^n n^6$ is convergent [12 Marks]
24. Let $f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x,y) = (0,0) \end{cases}$ Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(0,0)$ though $f(x,y)$ is not continuous at $(0,0)$. [15 Marks]
25. Find the minimum distance of the line given by the planes $3x + 4y + 5z = 7$ and $x - z = 9$ and from the origin, by the method of Lagrange's multipliers. [15 Marks]
26. Let $f(x)$ be differentiable on $[0,1]$ such that $f(1) = f(0) = 0$ and $\int_0^1 f^2(x) dx = 1$. Prove that $\int_0^1 xf(x)f'(x)dx = -\frac{1}{2}$ [15 Marks]
27. Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify your answer [20 Marks]

2011

28. Let $S = (0,1)$ and f be defined by $f(x) = \frac{1}{x}$ where $0 < x \leq 1$ (in \mathbb{R}). Is f uniformly continuous on S ? Justify your answer. [12 Marks]
29. Let $f_n(x) = nx(1-x)^n, x \in [0,1]$. Examine the uniform convergence of $\{f_n(x)\}$ on $[0,1]$ [15 Marks]
30. Find the shortest distance from the origin $(0,0)$ to the hyperbola $x^2 + 8xy + 7y^2 = 225$ [15 Marks]
31. Show that the series for which the sum of first n terms $f_n(x) = \frac{nx}{1+n^2x^2}, 0 \leq x \leq 1$ cannot be differentiated term-by-term at $x = 0$. What happens at $x \neq 0$? [15 Marks]
32. Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4x^2}$, then its derivative $S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$, for all x [20 Marks]

2010

33. Discuss the convergence of the sequence $\{x_n\}$ where $X_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{8}$ [12 Marks]
34. Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4+x_n}$ for $n > 1$ Show that the sequence converges to $\left(\frac{1+\sqrt{17}}{2}\right)$ [12 Marks]
35. Define the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Find $f'(x)$. Is $f'(x)$ continuous at $x = 0$? Justify your answer. [15 Marks]
36. Consider the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^2}$. Find the values of x for which it is convergent and also the sum function. Is the converse uniform? Justify your answer. [15 Marks]

37. Let $f_n(x) = x^n$ on $-1 < x \leq 1$ for $n = 1, 2, \dots$. Find the limit function. Is the convergence uniform? Justify your answer. [15 Marks]

2009

38. State Roll's theorem. Use it to prove that between two roots of $e^x \cos x = 1$ there will be a root of $e^x \sin x = 1$ 2+10=[12 Marks]

39. Let $f(x) = \begin{cases} \frac{|x|}{2} + 1 & \text{if } x < 1 \\ \frac{x}{2} + 1 & \text{if } 1 \leq x < 2 \\ -\frac{|x|}{2} + 1 & \text{if } 2 \leq x \end{cases}$ What are the points of discontinuity of f , if any? What are the points where f is not differentiable, if any? Justify your answer. [12 Marks]

40. Show that the series $\left(\frac{1}{3}\right)^2 + \left(\frac{1.4}{3.6}\right)^2 + \dots + \left(\frac{1.4.7 \dots (3n-2)}{3.6.9 \dots 3n}\right)^2 + \dots$ converges [15 Marks]

41. Show that if $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function then $f([a, b]) = [c, d]$ for some real numbers c and d , $c \leq d$.

[15 Marks]

42. Show that: $\lim_{x \rightarrow 1} \sum_{n=1}^{\infty} \frac{n^2 x^2}{n^4 + x^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1}$ Justify all steps of your answer by quoting the theorems you are using [15 Marks]

43. Show that a bounded infinite subset R must have a limit point [15 Marks]

2008

44. (i) For $x > 0$, show $\frac{x}{1+x} < \log(1+x) < x$ [6 Marks]

- (ii) Let $T = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \cup \left\{ 1 + \frac{3}{2n}, n \in \mathbb{N} \right\} \cup \left\{ 6 - \frac{1}{3n}, n \in \mathbb{N} \right\}$. Find derived set T' of T . Also find Supremum of T and greatest number of T . [6 Marks]

45. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $f(x+y) = f(x) + f(y)$, for all $x, y \in \mathbb{R}$ then show that $f(x) = xf(1)$ for all $x \in \mathbb{R}$. [12 Marks]

46. Discuss the convergence of the series $\frac{x}{2} + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \dots, x > 0$. [15 Marks]

47. Show that the series $\sum \frac{1}{n(n+1)}$ is equivalent to $\frac{1}{2} \prod_2^{\infty} \left(1 + \frac{1}{n^2 - 1} \right)$ [15 Marks]

48. Let f be a continuous function on $[0, 1]$. Using first Mean Value theorem on Integration, prove that

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{nf(x)}{1+n^2x^2} dx = \frac{\pi}{2} f(0) \quad [15 \text{ Marks}]$$

49. (i) Prove that the sets $A = [0, 1]$, $B = (0, 1)$ are equivalent sets. [6 Marks]

- (ii) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}$, $x \in \left(0, \frac{\pi}{2} \right)$ [9 Marks]

2007

50. Show that the function given by $f(x, y) = \begin{cases} \frac{xy}{x^2 + 2y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ is not continuous at $(0, 0)$ but its partial derivatives f_x and f_y exists at $(0, 0)$ [12 Marks]
51. Using Lagrange's mean value theorem, show that $|\cos b - \cos a| \leq |b - a|$ [12 Marks]
52. Given a positive real number a and any natural number n , prove that there exists one and only one positive real number ξ such that $\xi^n = a$ [20 Marks]
53. Find the volume of the solid in the first octant bounded by the paraboloid $z = 36 - 4x^2 - 9y^2$ [20 Marks]
54. Rearrange the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ to converge to 1 [20 Marks]

2006

55. Examine the convergence of $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$ [12 Marks]
56. Prove that the function f defined by $f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$ is nowhere continuous. [12 Marks]
57. A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is at least one value ξ , $a < \xi < b$ for which $f''(\xi) < 0$. [20 Marks]
58. Show that the function given by $f(x, y) = \begin{cases} \frac{x^3 + 2y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$ (i) is continuous at $(0, 0)$ (ii) possesses partial derivative $f_x(0, 0)$ and $f_y(0, 0)$ [20 Marks]
59. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ [20 Marks]

2005

60. If u, v, w are the roots of the equation in λ and $\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1$, evaluate $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ [12 Marks]
61. Evaluate $\iiint \ln(x+y+z) dx dy dz$ The integral being extended over all positive values of x, y, z such that $x+y+z \leq 1$ [12 Marks]
62. If f' and g' exist for every $x \in [a, b]$ and if $g'(x)$ does not vanish anywhere (a, b) , show that there exists c in (a, b) such that $\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$ [30 Marks]
63. Show that $\int_0^{\infty} e^{-t} t^{n-1} dt$ is an improper integral which converges for $n > 0$ [30 Marks]

2004

64. Show that the function $f(x)$ defined as: $f(x) = \frac{1}{2^n}$, $\frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}$, $n = 0, 1, 2, \dots$ and $f(0) = 0$ is integrable in $[0, 1]$, although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x) dx = \frac{2}{3}$ [12 Marks]
65. Show that the function $f(x)$ defined on by: $f(x) = \begin{cases} x & \text{when } x \text{ is irrational} \\ -x & \text{when } x \text{ is rational} \end{cases}$ is continuous only at $x = 0$ [12 Marks]
66. If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c . [20 Marks]
67. Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$ [20 Marks]
68. Let $f(x) \geq g(x)$ for every x in $[a, b]$ and f and g are both bounded and Riemann integrable on $[a, b]$. At a point $c \in [a, b]$, let f and g be continuous and $f(c) > g(c)$ then prove that $\int_a^b f(x) dx > \int_a^b g(x) dx$ and hence show that $-\frac{1}{2} < \int_a^b \frac{x^3 \cos 5x}{2+x^2} dx < \frac{1}{2}$ [20 Marks]

2003

69. Let α be a positive real number and $\{x_n\}$ sequence of rational numbers such that $\lim_{n \rightarrow \infty} x_n = 0$. Show that $\lim_{n \rightarrow \infty} \alpha x_n = 1$ [12 Marks]
70. If a continuous function of x satisfies the functional equation $f(x+y) = f(x) + f(y)$ then show that $f(x) = \alpha x$ where α is a constant. [12 Marks]
71. Show that the maximum value of $x^2 y^2 z^2$ subject to condition $x^2 + y^2 + z^2 = c^2$ is $\frac{c^2}{27}$. Interpret the result [20 Marks]
72. The axes of two equal cylinders intersect at right angles. If α be their radius, then find the volume common to the cylinder by the method of multiple integrals. [20 Marks]
73. Show that $\int_0^{\infty} \frac{dx}{1+x^2 \sin^2 x}$ is divergent [20 Marks]

2002

74. Prove that the integral $\int_0^{\infty} x^{m-1} e^{-x} dx$ is convergent if and only if $m > 0$. [12 Marks]
75. Find all the positive values of a for which the series $\sum_{n=1}^{\infty} \frac{(an)^n}{n!}$ converges. [12 Marks]
76. Test uniform convergence of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$, where $p > 0$ [20 Marks]
77. Obtain the maxima and minima of $x^2 + y^2 + z^2 - yz - zx - xy$ subject to condition $x^2 + y^2 + z^2 - 2x + 2y + 6z + 9 = 0$ [25 Marks]

78. A solid hemisphere H of radius ' a ' has density ρ depending on the distance R from the center of and is given by $\rho = k(2a - R)$ where k is a constant. Find the mass of the hemisphere by the method of multiple integrals [15 Marks]

2001

79. Show that $\int_0^{\pi/2} \frac{x^n}{\sin^m x} dx$ exists if and only if $m < n + 1$ [12 Marks]
80. If $\lim_{n \rightarrow \infty} a_n = l$, then prove that $\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = l$, [12 Marks]
81. A function f is defined in the interval (a, b) as follows

$$f(x) = \begin{cases} \frac{1}{q^2} & \text{when } x = \frac{p}{q} \\ \frac{1}{q^3} & \text{when } x = \sqrt{\frac{p}{q}} \end{cases} \text{ where } p, q \text{ relatively prime integers. } f(x) = 0 \text{ for all other values of } x. \text{ Is } f$$

- Riemann integrable? Justify your answer. [20 Marks]
82. Show that $U = xy + yz + zx$ has a maximum value when the three variables are connected by the relation $ax + by + cz = 1$ and a, b, c are positive constants satisfying the condition $2(ab + bc + ca) > (a^2 + b^2 + c^2)$ [25 Marks]
83. Evaluate $\iiint (ax^2 + by^2 + cz^2) dx dy dz$ taken throughout the region $x^2 + y^2 + z^2 \leq R^2$ [15 Marks]

2000

84. Given that the terms of a sequence $\{a_n\}$ are such that each $a_k, k \leq 3$, is the arithmetic mean of its two immediately preceding terms. Show that the sequence converges. Also find the limit of the sequence. [12 Marks]
85. Determine the values of x for which the infinite product $\prod_{n=0}^{\infty} \left(1 + \frac{1}{x^{2n}}\right)$ converges absolutely. Find its value whenever it converges. [12 Marks]
86. Suppose f is twice differentiable real valued function in $(0, \infty)$ and M_0, M_1 and M_2 the least upper bounds of $|f(x)|, |f'(x)|$ and $|f''(x)|$ respectively in $(0, \infty)$. Prove for each $x > 0, h > 0$ that $f'(x) \frac{1}{2h} [f(x+2h) - f(x)] - hf'(u)$ for some $u \in (x, x+2h)$. Hence show that $M_1^2 \leq 4M_0M_2$. [20 Marks]
87. Evaluate $\iint_S (x^3 dy dz + x^2 y dz dx + x^2 z dx dy)$ by transforming into triple integral where S is the closed surface formed by the cylinder $x^2 + y^2 = a^2, 0 \leq z \leq b$ and the circular disc $x^2 + y^2 \leq a^2, z = 0$ and $x^2 + y^2 \leq a^2, z = b$ [20 Marks]

1999

88. Let A be a subset of the metric space (M, ρ) . If (A, ρ) is compact, then show that A is a closed subset of (M, ρ) [20 Marks]
89. A sequence $\{S_n\}$ is defined by the recursion formula $S_{n+1} = \sqrt{3S_n}, S_1 = 1$. Does this sequence converge? If so, find $\lim S_n$ [20 Marks]
90. Test for convergence the integral $\int_0^1 x^p \left(\log \frac{1}{x}\right)^q dx$ [20 Marks]

91. Find the shortest distance from the origin to the hyperbola $x^2 + 8xy + 7y^2 = 225, z = 0$ [20 Marks]
92. Show that the double integral $\iint_R \frac{x-y}{(x+y)^3} dx dy$ does not exist over $R = [0,1;0,1]$ [20 Marks]
93. Verify the Gauss divergence theorem for $\vec{F} = 4x\hat{e}_x - 2y^2\hat{e}_y + z^2\hat{e}_z$ taken over the region bounded by $x^2 + y^2 = 4, z = 0$ and $z = 3$ where $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are unit vectors along $x-, y-$ and $z-$ directions respectively. [20 Marks]

1998

94. Let X be a metric space and $E \subset X$. Show that
 (i) Interior of E is the largest open set contained in E
 (ii) Boundary of $E = (\text{closure of } E) \cap (\text{closure of } X - E)$ [20 Marks]
95. Let (X, d) and (Y, e) be metric spaces with X compact and $f : X \rightarrow Y$ continuous. Show that f is uniformly continuous. [20 Marks]
96. Show that the function $f(x, y) = 2x^4 - 3x^2y + y^2$ has $(0, 0)$ as the only critical point but the function neither has a minima nor maxima at $(0, 0)$ [20 Marks]
97. Test the convergence of the integral $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx, a \geq 0$ [20 Marks]
98. Test the series $\sum_{n=1}^\infty \frac{x}{(n+x^2)^2}$ for uniform convergence. [20 Marks]
99. Let $f(x) = x$ and $g(x) = x^2$. Does $\int_0^1 f \circ g$ exist? If it exists then find its value [20 Marks]

1997

100. Show that a non-empty set P in R^n each of whose points is a limit-point is uncountable. [20 Marks]
101. Show that $\iiint_D xyz dx dy dz = \frac{a^2 b^2 c^2}{6}$ where domain D is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ [20 Marks]
102. If $u = \sin^{-1} \left[(x^2 + y^2)^{1/5} \right]$, Prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2}{25} \tan u (2 \tan^2 u - 3)$ [20 Marks]

1996

103. Let F be the set of all real valued bounded continuous functions defined on the closed interval $[0, 1]$. Let d be a mapping of $F \times F$ into R , the set of real numbers, defined by $d(f, g) = \int_0^1 |f(x) - g(x)| dx \quad \forall f, g \in F$.
 Verify that d is a metric for F [20 Marks]
104. Prove that a compact set in a metric space is a closed set. [20 Marks]
105. Let $C[a, b]$ denote the set of all functions f on $[a, b]$ which have continuous derivatives at all points of $I = [a, b]$. For $f, g \in C[a, b]$ define $d(f, g) = |f(a) - g(b)| + \sup \{ |f'(x) - g'(x)|, x \in I \}$. Show that the space

$(C[\alpha, b], d)$ is a complete

[20 Marks]

106. A function f is defined in the interval (α, b) as follows:

$$f(x) = \begin{cases} q^{-2} & \text{when } x = pq^{-1} \\ q^{-3} & \text{when } x = (pq^{-1})^{1/2} \end{cases}$$

where p, q are relatively prime integers; $f(x) = 0$, for all other values of x . Is f Riemann integrable? Justify your answer. [20 Marks]

107. Test for uniform convergence, the series $\sum_{n=1}^{\infty} \frac{2^n x^{2n-1}}{1+x^{2n}}$ [20 Marks]

108. Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y) dx dy$ [20 Marks]

1995

1. Let K and F be nonempty disjoint closed subsets of R^2 . If K is bounded, show that there exists $\delta > 0$ such that $d(x, y) \geq \delta$ for $x \in K$ and $y \in F$ where $d(x, y)$ is the usual distance between x and y . [20 Marks]

2. Let f be a continuous real function on R such that f maps open interval into open intervals. Prove that f is monotonic. [20 Marks]

3. Let $c_n \geq 0$ for all positive integers n such that $\sum c_n$ is convergent. Suppose $\{S_n\}$ is a sequence of distinct points in (a, b) . For $x \in [a, b]$, define $\alpha(x) = \sum c_n \chi_{\{x > S_n\}}$. Prove that α is an increasing function. If f a continuous

real function on $[a, b]$, show that $\int_a^b f d\alpha = \sum c_n f(S_n)$ [20 Marks]

4. Suppose f maps an open ball $U \subset R^n$ into R^m and f is differentiable on U . Suppose there exists a real number $M > 0$ such that $\|f'(x)\| \leq M \forall x \in U$. Prove that $|f(b) - f(a)| \leq M|b - a| \forall a, b \in U$ [20 Marks]

5. Find and classify the extreme values of the function $f(x, y) = x^2 + y^2 + x + y + xy$ [20 Marks]

6. Suppose α is real number not equals to $n\pi$ for any integer n . Prove that $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+2xy \cos \alpha + y^2)} dx dy = \frac{\alpha}{2 \sin \alpha}$ [20 Marks]

1994

7. Examine the (i) absolute convergence (ii) uniform convergence of the series $(1-x) + x(1-x) + x^2(1-x) + \dots$ in $[-c, 1]$, $0 < c < 1$ [20 Marks]

8. Prove that $S(x) = \sum \frac{1}{n^p + n^q x^2}$, $p > 1$ is uniformly convergent for all values of x and can be differentiate term by term if $q < 3p < 2$ [20 Marks]

9. Let the function f be defined on $[0, 1]$ by the condition $f(x) = 2rx$ when $\frac{1}{r+1} < x < \frac{1}{r}$, $r > 0$ Show that f is

Riemann integrable in $[0, 1]$ and $\int_0^1 f(x) dx = \frac{\pi^2}{6}$ [20 Marks]

10. By means of substitution $x + y + z = u, y + z = uv, z = uvw$ evaluate $\iiint (x + y + z)^n xyz \, dx dy dz$ taken over the volume bounded by $x = 0, y = 0, z = 0, x + y + z = 1$ [20 Marks]

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11. Examine for Riemann integrability over $[0, 2]$ of the function defined in $[0, 2]$ by

$$f(x) = \begin{cases} x + x^2, & \text{for rational values of } x \\ x^2 + x^3, & \text{for irrational values of } x \end{cases} \quad [20 \text{ Marks}]$$

12. Prove that $\int_0^{\infty} \frac{\sin x}{x} \, dx$ converges and conditionally converges. [20 Marks]

13. Evaluate $\iiint \frac{dx dy dz}{x + y + z + 1}$ over the volume bounded by the coordinate planes and the plane $x + y + z = 1$ [20 Marks]

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14. If we metrize the space of functions continuous on $[a, b]$ by taking $p(x, y) = \sqrt{\int_a^b [x(t) - y(t)]^2 \, dt}$ then show that the resulting metric space is NOT complete [20 Marks]

15. Examine $2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y - 4z$ for extreme values [20 Marks]

16. If $U_n = \frac{1 + nx}{ne^{nx}} - \frac{1 + (n+1)x}{(n+1)e^{(n+1)x}}$, $0 < x < 1$ Prove that $\frac{d}{dx} \sum U_n = \sum \frac{d}{dx} U_n$ Is the series uniformly convergent in $(0, 1)$? Justify your claim. [20 Marks]

17. Find the upper and lower Riemann integral for the function defined in the interval $[0, 1]$ as follows

$$\begin{cases} \sqrt{1-x^2} & \text{when } x \text{ is rational} \\ 1-x & \text{when } x \text{ is irrational} \end{cases} \quad \text{and show that is NOT Riemann integrable in } [0, 1]. \quad [20 \text{ Marks}]$$

18. Discuss the convergence or divergence of $\int_0^{\infty} \frac{x^\beta}{1 + x^\alpha \sin^2 x} \, dx$, $\alpha > \beta > 0$ [20 Marks]

19. Evaluate $\iint \frac{\sqrt{a^2 b^2 - b^2 x^2 - a^2 y^2}}{\sqrt{a^2 b^2 + b^2 x^2 + a^2 y^2}} \, dx dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [20 Marks]