Linear Algebra

Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

i.e $\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$

i.e $6 - 2\lambda - 3\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 5\lambda + 4 = 0 \Rightarrow (\lambda - 4)(\lambda - 1) = 0 \Rightarrow \lambda = 4, 1$

$\therefore$ The eigenvalues of A are 4, 1

The eigenvectors of $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of A corresponding to eigenvalue of '4' are given by the non-zero solution of the equation $(A - 4I)X = 0$

$$\begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R_2 \to R_2 + \tfrac{1}{2}R_1$$

$-2x_1 + 2x_2 = 0 \Rightarrow x_1 = x_2$

Let us take $x_1 = 1, x_2 = 1$, then $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Similarly The eigenvectors X of A corresponding to eigenvalue '1' are given by the non-zero solution of the

$(A - \lambda I)X = 0 \Rightarrow (A - I)X = 0$

$$\Rightarrow \begin{bmatrix} 2-1 & 2 \\ 1 & 3-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad R_2 \to R_2 - R_1$$

$$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$$

let us take $x_2 = 1$, $x_1 = -2$, then $X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now P: Transformation matrix $= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

And we have to show that $P^{-1}AP = D$

$$\Rightarrow \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = D$$

$\Rightarrow$ diagonal matrix $= \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ //

Show that similar matrices have same characteristic polynomial.

Sol: let A and B be two similar matrices & $P(t)$ denote k^{th} degree polynomial.

To show: $P(A)$ and $P(B)$ are similar.

$\because$ matrices A and B are similar. So,

$B = P^{-1}AP$, for some $(n \times n)$ invertible matrix P.

Consider

$$\det(B) = \det(P^{-1}AP)$$

$$= \det(P^{-1}) \det(A) \det(P)$$

$$= \det(A) \det(P^{-1}) \det(P)$$

$$= \det(A) \det(P^{-1}P) \qquad (\because I = P^{-1}P$$

$$= \det(A) \det(I) = \det(A)$$

$$=$$

Now, $P(B) = |B - \lambda I|$

consider,

$B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - \lambda(P^{-1}IP)$

$= P^{-1}(A - \lambda I)P$

$\therefore$ determinant of $B - \lambda I$ is same as $A - \lambda I$

$\therefore$ $P(A)$ and $P(B)$ are similar.

Hence, the similar matrices have the same characterstic polynomial.

Suppose U and W are distinct four dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimension of a subspace $U \cap W$.

Sol: because U and W are distinct, $U + W$ properly contains U and W;

Consequently $\dim(U+W) > 4$

but $\dim(U+W)$ cannot be greater than 6, as $\dim V = 6$.

Hence we have '2' possibilities.

(a) $\dim(U+W) = 5$, (b) $\dim(U+W) = 6$

We know that

$$\dim(U \cap W) = \dim U + \dim W - \dim(U+W)$$

$$= 8 - \dim(U+W)$$

thus

(a) $\dim(U \cap W) = 3$ (or) (b) $\dim(U \cap W) = 2$

Consider the matrix mapping $A: R^4 \to R^3$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$$, find a basis and dimension of the image of A and those of kernel A.

Sol:

(a) The column space of A is equal to Im A, Now reduce A^T to echelon form.

$$A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 8 \\ 3 & 5 & 13 \\ 1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

thus $\{(1,1,3)\ (0,1,2)\}$ is a basis of Im A, and

$\dim(\text{Im } A) = 2$ //

(b) Here ker A is the solution space of the homogeneous system of $AX = 0$, where $X = \{x, y, z, t\}^T$, thus reduce the matrix A of coeffecients to echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(or) $x + 2y + 3z + t = 0$

$y + 2z - 3t = 0$

the free variables are z and t, thus $\dim(\ker A) = 2$

(i) set $z = 1,\ t = 0$ to get the solution $(1, -2, 1, 0)$

(ii) set $z = 0,\ t = 1$ to get the solution $(-7, 3, 0, 1)$

thus $(1,-2,1,0)$ & $(-7,3,0,1)$ form a basis of ker A.